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THE BAKSHĀLĪ MANUSCRIPT

BY

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§ 1. Introductory

In spite of the publication by the Archæological Survey of India, of the text of the Bakshālī Manuscript † with elaborate introduction and notes by G. R. Kaye, there are perhaps many fresh points tending to throw additional light on the contents and the age of the manuscript. A considerable portion of the analysis of the manuscript was really due to Hoernle, and by an irony of fate, the work was left for completion to one whose views and prejudices were poles apart from those of the pioneer. It is regretted that Kaye's examination apparently scientific, is warped by his profound obsession that he should find traces of Greek influence in everything Indian. This attitude has vitiated most of his conclusions and made it worth while for others * to re-examine his statements and give them their proper weight. The present paper is an attempt at such re-examination and re-assessment.

It is now sufficiently known that a manuscript of 70 leaves of birch-bark, each 7″ by 4″, dealing mainly with Arithmetic was discovered in 1881 by a peasant, near a village called Bakshali about 70 miles from the famous Taxila in the north-western frontier of India. This manuscript has since been christened "The Bakshālī Manuscript." The text is written in Sarada script the date of which is highly disputed and the language is the so-called *Gatha* dialect a form of north-western *Prakrit.* Apart from the script and the language, there are many surprises in the mathematical contents

† The Bakshālī Manuscript by G. R. Kaye, *Archæological Survey of India,* New Imperial Series, Vol. XLIII, Parts I and II (1927) and Part III (1933). Hereafter referred to as "B. M. I, II, III."

* For example, *The Bakshālī Mathematics,* by Bibhutibhusan Datta, 1929; hereafter referred to as "Datta."

which make one hesitate in fixing its age. These surprises and peculiarities will now be studied under seven heads:

(i) The method of presentation,

(ii) Peculiar terminology, abbreviations, and the cross-symbol for subtraction,

(iii) The Decimal Notation and the absence of word-numerals,

(iv) The symbol for the unknown,

(v) The Rule of *Regula Falsi*,

(vi) The square-root rule and the process of reconciliation,

and (vii) Series and sequences.

§ 2. The Method of Presentation

Rules and examples are presented in verse in Sloka metre and the explanations are given in prose according to a certain stereotyped convention as follows:

(1) The re-statement of the problem in a symbolic notation–called Sthāpana or Nyāsa.

(2) The working proper—called Karaṇa

and (3) Verification—called Pratyaya.

This bears some analogy to the Euclidean scheme of

(1) particular enunciation corresponding to Nyāsa,

(2) construction corresponding to Karaṇa,

and (3) demonstration corresponding to verification.

It is interesting to note that among the early arithmeticians verification had played an important part and served almost as an equivalent of 'logical proof'. The majority of minds are content with the working rule, provided verification shows them that the results are correct. They do not care to worry themselves about the rationale. In actual teaching again, we find a pupil more easily convinced by verification than by a closely reasoned mathematical argument too difficult to follow. Thus, verification is a potent instrument in the pedagogy of mathematics. No wonder, therefore, that a mathematics teacher of bygone days should have thought of incorporating 'verification' as an act of faith in a work intended especially for the benefit of young pupils †. In the Bakshālī Manuscript we have perhaps a glimpse of a sort of teaching notes—something intermediary to an original treatise and a regular commentary—of a private tutor. The loose colloquial style adopted is also in keeping

† Vide the Colophon. B. M. II 50, recto. P. 142.

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with this idea. One may also recollect in this connection that Bakshālī was near the famous ancient University centre Taxila.

'Verification' was common enough among the Indian mathematical commentators as early as the time of Chaturveda Prithūdakaswamí of the 9th century A. D. The rule of Vyastavidhi or reversing the steps, discussed in almost all ancient Indian mathematical works is perhaps intended to be useful in 'verification', especially of the roots of equations and in the so-called 'think of a number' problems. For example,* we have the result $1+\frac{1}{2}\{2+\frac{1}{2}(3+\frac{2}{2}\cdot\overline{4+\frac{3}{2}\cdot\overline{5+\frac{4}{2}\cdot\frac{1}{4}}})\}=\frac{93}{16}$ verified by the principle of reversal of steps, viz.

$$\frac{2}{4}\left[\frac{2}{3}\left[\frac{2}{2}\left[2\left\{2\left(\frac{93}{16}-1\right)-2\right\}-3\right]-4\right]-5\right]=\frac{1}{4},$$ worked up from the innermost bracket outwards step by step.

§ 3. Peculiar Terminology etc.

The mathematical terminology adopted in the Bakshālī Manuscript though generally the same as in other Hindu Mathematical works, contains a few exceptions incidental more or less to the language or dialect employed. For example, the terms such as 'Partha', 'Dhanta', 'Pasta' are peculiar to the Gatha dialect. The use of the terms 'Savarna', 'Kalāsavarna', and 'Sadris'ikarana' with reference to fractions remind one of similar uses in other Hindu works.

There are several abbreviations, some of which are consistently employed, while others are used without any uniformity. For example, the abbreviation *yu* for *yutam* (for addition) is sometimes put in between and sometimes after the expressions to be summed:

** e.g.

°	5	*yu*°
1	1	

and

11	*yu*°	5
1		1

meaning respectively $x+5$ and $11+5$.

The abbreviation *gu* for guṇitam (multiplication) is sometimes put in but very often dropped:

*** e.g.

2	5		3	*gu*°	7		4	*gu* · · · · · ·
1	2†		1		2†		1	

The text is mutilated here but it is clear that *gu* is omitted between $\frac{2}{1}$ and $\frac{5}{2}$†.

* B. M. III pp. 187, 188; 67 verso and recto.

** B. M. III Folio 59 recto; p. 215.

*** B. M. III Folio 25 verso; p. 196, last line above the foot-note.

It is likely that many abbreviations are 'ad-hoc' inventions of the scribes themselves, who must belong to a period much later than the original author and one need not therefore attach to them any historical or mathematical importance.

There is one unique symbol whose use is fairly consistent but for stray exceptions evidently due to the carelessness of the scribes. The sign of the cross (†) is used to denote Riṇa or negative and placed to the right of the number which it qualifies. The symbol is peculiar to the Bakshālī Manuscript and does not occur elsewhere in Hindu mathematical works. It is also an operational symbol for subtraction. In the Nyasa for a problem on profit and loss,* we have (†) after 56 in

16	6	riṇam 56†
5	1	1

to indicate that 56 is the loss. Again in another, ** we have

1	2	3	6
2	5	·	4†

where the significance of the symbol † is not clear. Apart from these two anomalies, wherever the cross † is used, the number preceding it is meant to be subtracted. On account of some resemblance (suggested first by Thibaut) between this cross-mark and the Diophantine mark ↑ used for a similar purpose but in a different position, Kaye is inclined to perceive here a Greek influence; while Hoernle suggests that the Indian symbol may be an abbreviation of one of the numerous words riṇa, kanita, ūna, kshaya meaning diminution and makes a particular reference to *ka*, written exactly like the Bakshālī cross-symbol in the Asoka script. But Kaye gravely points out the danger of tracing an isolated symbol back through the ages, while he himself complacently falls into the same error with respect to the Diophantine symbol. In this connection we may recollect D. E. Smith's[4] warning that the Arithmetic of Diophantus now extant was written in the 13th century (a thousand years after the original) and we have to allow for the possible interpolations of medieval copyists. In the early medieval period in India, say from the time of Bhaskara, the dot-symbol (·) for the negative became the fashion. In a subsequent section we discuss two interesting uses of the dot-symbol in the Bakshālī Manuscript. The absence of the use of the dot-symbol for negative quantities is an important evidence that the Bakshālī work must have preceded Bhaskara.

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* B. M. III p, 220. 63 recto. ** B. M. III 18 recto. p. 211.

§ 4. The Decimal Notation and the absence of Word-numerals

Throughout the Bakshālī Manuscript, some special numerals are used and the decimal notation employed in much the same way as in the modern notation. There is no attempt made to mention big numbers in words. The only other early Indian work systematically avoiding word-numerals is the[5] Aryabhatīya containing an ad-hoc notation for expressing big numbers. The fashion of word-numeration, i.e., using a word to connote the idea of a number, say 'eyes' for '2', 'tithi' (lunar days of the half month) for '15', 'teeth' for '32' and so on, was set by Varāhamihira in the sixth century and since then became an extraordinarily popular method of numeration in Hindu mathematical and other works. The systematic elimination of word-numerals in the present work is a sure index that it belongs to a period when the word-numeration had not yet become the general fashion. The employment of decimal notation in the present work is an additional evidence that the notation flourished in India even in the early centuries of the Christian era. But Kaye argues the other way about and ascribes the work to the medieval period on account of the decimal notation. Every Hindu work from the earliest times including the Vedic period shows intimate familiarity with the decimal numeration, one, ten, hundred, thousand, etc. which naturally led sooner or later unmistakably to the decimal notation with its ten figures and place-value. The play of position is a conspicuous feature of early Indian arithmetical symbolism.[(2)]

§ 5. The Symbol for the Unknown

In the Bakshālī Manuscript the dot (·) is used with apparently two different kinds of significance but with the same underlying idea of 'void', 'gap', or 'emptiness'. The dot, primarily a symbol of 'emptiness' must have become secondarily a symbol for the unknown or absent quantity. For, in one of the Sūtras* we are directed to put any desired number in the place of the unknown marked by the sign of 'emptiness' (Sūnya). An analogous use[(3)] of the zero for the unknown quantity in a proportion appears in a Latin manuscript of some lectures by Gottfried Wolack in the University of Erfont in 1467 and 1468. When the Sūnya stands for the unknown, a coefficient 1 is always set under it, as for example, $\binom{y\bar{a}}{1}$ in Bhaskara's Bijaganita. Another curious feature of the Bakshālī dot-mark is that it does duty simultaneously for several

* B. M. II. Folio 22 verso and 23 recto, pp. 122, 123.

unknowns, probably because it was felt to be more general and abstract than a literal or any other symbol that might be thought of:

e. g.

·	5	*yu*	*mu*	·	sā	·	7†	*mu*	·
1	1			1		1	1		1

which means $\sqrt{x+5} = y, \quad \sqrt{x-7} = z.$

Kaye believes that the ambiguous use of the dot-symbol is an indication of the lack of an efficient symbolism and is perplexed by the denominator unity for the dot-symbol. He fails to appreciate that the denominator unity serves to indicate that the dot-symbol is a symbol for the unknown and not the symbol for 'zero'. Here we have again another hint that the contents of the manuscript must belong to a period when the literal notation for the unknown did not come into general use in Hindu works.

The symbol for 'zero', the negative sign, and the symbol for the unknown seem to have a common ancestry and their nebulous beginnings are perhaps reflected in the Bakshālī Manuscript.

§ 6. The Rule of Regula Falsi

David Eugene Smith remarks[4] 'Awkward as this seems, the rule (explained below) was used for many centuries, a witness to the need for and value of a good symbolism'. Kaye believes in this dictum. The rule of double false to solve $ax+b=0$, gives

$$x = \frac{f_1 g_2 - f_2 g_1}{f_1 - f_2} \text{ where } ag_1 + b = f_1 \text{ and } ag_2 + b = f_2, \quad f_1, f_2$$

being called the two false values or failures corresponding to g_1 and g_2 respectively. In effect, I should say that this method is the same as that in analytical geometry to find the co-ordinates of the point where the line joining (g_1, f_1) and (g_2, f_2) meets the x-axis. From another point of view, the method may be deemed to involve the principle of linear interpolation. After all, does not the theory of interpolation involve the introduction of not two, but several falses hovering about the true value one is in quest of? When the truth happens to lie deep hidden, we are sometimes forced to accept the nearest falsehood as truth. Hence the method of the double false is not an awkward one but envisages an essential process of making falses themselves yield truth, a precursor of modern interpolation theory.

But the above rule ought not to be confused with the Indian Ishṭakarma which is an operation with an assumed number, used in

cases where the final result arrived at after a series of manipulations is always proportional to the number originally assumed. The required number can be obtained by Trairāśika or proportion corresponding to the given final result or Driṣya. Algebraically, the Ishṭakarma is equivalent to solving the simple equation $ax=b$ and arriving at $x=\frac{bm}{am}$, where am is the value of the left-hand side when $x=m$ (Ishṭa). It is important to note that Ishtakarma is a useful device to evaluate x when a is a complicated expression, say $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})$. When a is a straightforward simple coefficient, the Hindu mathematician would straightaway utilise his Rule of Inverse Operations and get the value of x. The Ishṭakarma must be regarded therefore more as an appendix to Trairāśika (proportion) than as a degenerate case of 'Regula Falsi'.

The use of Regula Falsi in the Bakshālī Manuscript as conceived by Kaye† is not the same as the one used in the middle ages but a fanciful deduction based on a misunderstanding of an ingenious method of generalised Ishṭakarma to be presently explained. The author of Bakshālī Manuscript solves the equations

$$x_1+x_2=16,\ x_2+x_3=17,\ x_3+x_4=18,\ x_4+x_5=19,\ x_5+x_1=20$$

by assuming a tentative value (Ishṭa), say 7 for x_1 and derives from it successively the values, 9, 8, 10, 9 for x_2, x_3, x_4, x_5, and thence $x_5+x_1=16$. But from the structure of the equations, it is noticed that any decrease or increase in x_1 involves a corresponding and same decrease or increase in x_3 and x_5, so that the decrease or increase gets doubled for x_1+x_5. Hence the true value for x_1 can be obtained from the tentative value by decreasing or increasing it by *half* the total deficit or excess in the actual given value of x_1+x_5 as compared with the value based on our assumption. Thus the assumed value 7 should be increased by 2 and $x_1=9$. Thence $x_2=7$, $x_3=10$, $x_4=8$, $x_5=11$.

This method again is peculiar to Bakshâlî Mathematics and I have not come across the like of it elsewhere. To trace this to the medieval Regula Falsi and thence infer an evidence that the work belongs to medieval times is a far-fetched argument.

The Hindu mathematician has never used the medieval 'Regula Falsi' to solve a simple equation of the type $ax+b=cx+d$. Kaye ought to have been aware of [5] Aryabhata's solution of this in the

† Vide B. M. I pp. 32, 33.

form $(d-b)/(a-c)$, as well as similar solutions presented in the text of Bakshālī Manuscript† and yet asserts that the Regula Falsi was used by the early Hindu mathematicians on account of lack of efficient symbolism. Regula Falsi as interpreted by Kaye, at least with reference to the Bakshālī Manuscript is regular falsehood.

§ 7. The Square-root rule and the Process of Reconciliation

Another striking feature in the mathematical contents of the Bakshālī Manuscript is the chapter on the square-root rule, where we have

$$\sqrt{a^2+\gamma} = a + \frac{\gamma}{2a} - \frac{(\gamma/2a)^2}{2\,(a+\gamma/2a)}$$

an approximation found in exactly the same form also in an Arab work of the twelfth century. This has given rise to a speculation that the present work may therefore be later than the twelfth century. But it must be noted that the rule is merely a corollary of the general square-root rule found explicitly in all Indian mathematical works, at any rate from Aryabhata onwards. The earliest evidence of a concrete numerical application of this rule even to a higher order than the one indicated by the above formula is to be found in the Sulva Sūtras, which contain the approximation $1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34}$ for $\sqrt{2}$. This formula can be easily explained by the usual Indian algorithm for finding square-roots, (the same as the one adopted even to-day):

$$\begin{array}{r|l}
 & \begin{matrix}2\\ 1\end{matrix} \;\Big|\; 1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34} \\
2+\frac{1}{3} & 1 \\
 & \underline{7/9} \\
 & 2/9 \\
2+\frac{2}{3}+\frac{1}{3\cdot4} & 2/9+\frac{1}{3\cdot4\cdot3\cdot4} \\
 & \overline{\qquad\qquad} \\
2+\frac{2}{3}+\frac{1}{3\cdot2}-\frac{1}{3\cdot4\cdot34} & -\frac{1}{3\cdot4\cdot3\cdot4} \\
 & -\frac{1}{3\cdot4\cdot3\cdot4}+\frac{1}{3\cdot4\cdot34\cdot3\cdot4\cdot34} \\
\hline
\end{array}$$

and so on, ad libitum.

Further, it is important to note that the Bakshâlī rule ought not to be so narrowly interpreted that it gives only the second approximation but is suggestive of a process which permits repeated application to

† B. M. III p. 173 (9 verso).

any desired order of approximation. Brahmagupta[6] mentions a similar rule in connection with the square-root of a number in sexagesimal notation.

The occasion for finding the square-root approximation is a very fanciful one. For, it is used to find the number of terms 't' of an arithmetical progression, when the sum s, the first term a and the common difference d are given. t is given by the formula:

$$t = \frac{d - 2a + \sqrt{(2a - d)^2 + 8ds}}{2d}$$

while
$$s = t\left(a + \overline{t-1}.\frac{d}{2}\right)$$

Both these occur in all early Hindu mathematical works. The new feature in the present work is the following:

If an approximation to $\sqrt{(2a-d)^2+8ds}$ be q_1 and the corresponding value of t be t_1 and the corresponding sum s_1, the verification of the approximation consists in noting that

$$s = s_1 - \frac{e_1}{8d} \text{ where } e_1 = q_1^2 - \{(2a-d)^2 + 8ds\}$$

The proof of this result is extremely simple, but Kaye puts up a long rigmarole.

In fact, if $s_1 = t_1\left(a + \overline{t_1 - 1}.\frac{d}{2}\right)$, it follows immediately

that
$$t_1 = \frac{d - 2a + \sqrt{(2a-d)^2 + 8ds_1}}{2d},$$

and hence
$$q_1^2 = (2a-d)^2 + 8ds_1$$
$$= (2a-d)^2 + 8ds + 8d\,(s_1 - s);$$
$$\therefore \quad e_1 = 8d\,(s_1 - s) \text{ or } s = s_1 - e_1/8d.$$

This method of verification is obviously applicable whatever be the degree of approximation. It is nothing peculiarly adapted to the approximation of the second degree. But the value of e_1 is either $\left(\frac{\gamma}{2a}\right)^2$ or $\frac{(\gamma/2a)^4}{4(a+\gamma/2a)^2}$ according as the first or second approximation is taken.

The author of the Bakshālī manuscript seems to take a peculiar pleasure in making huge calculations to verify

$$e_1 = 8d(s_1 - s)$$

in a number of examples. Mathematically, all this work is futile and meaningless, since the summation of an Arithmetical Progression to a non-integral number of terms is not practical mathematics. But such a summation seems to be very common in the days of Chaturveda, Sridhara and Mahavira, i.e. roughly the ninth and tenth centuries. Mahavira triumphantly asks "Give out the first term and the common difference respectively in relation to two series, in which $\frac{31}{150}$ is the sum, while $\frac{3}{4}$ in one case is the common difference, and $\frac{4}{5}$ the number of terms and in the other case $\frac{1}{3}$ the first term and $\frac{4}{5}$ the number of terms."(7) Very likely the Bakshali arithmetician who revels in summation of series to an irrational number of terms belongs to the same school and is perhaps contemporaneous with the writers just quoted. A typical Bakshālī problem involving square-root approximation may be given by way of illustration: 'A certain person goes 5 yojanas on the first day, and 3 yojanas more on each succeeding day. Another who travels 7 yojanas per day has a start of 5 days. When will they meet, say O! the best of mathematicians.'† The answer given in the text is $\frac{7+\sqrt{889}}{6}$ days or $6\frac{4}{29}$ days approximately, i.e. 6 days, 8 ghatikas,* 16 vighatis, 33^{iii}, $6^{iv}\frac{6}{29}$. The futility of the result transcends one's imagination. But Dr. Bibhutibhushan Datta(2) comments:

"The expression for the time in which two persons will meet contains a surd quantity. So the two persons will never meet"!! A worthy comment! The problem is quite sensible but not the above solution. The mathematician has strayed hopelessly away from common sense, carried away by his unqualified faith in the algebraic formula. The correct answer is 6 days, 7 ghatikas and 30 vighatis, quite rational and free from surds. The reader can easily verify this result.

It takes perhaps centuries to outgrow such impossible fantasies as those exhibited in the problems on square-root approximations. We are glad to find Bhaskara of the twelfth century entirely free from them. He was, perhaps, the earliest Hindu mathematician to perceive the beauty and importance of integers and integral solutions. His marvellous chakravāla method, whose import is even to this day imperfectly understood, of solving $x^2 - Ny^2 = 1$ (N a non-square positive integer) in positive integers is a remarkable instance of 'the sense' for the integer. The Bakshālī mathematician is too ancient to have cultivated this 'sense.'

† B. M. III p. 178. 6 recto.

* 1 Ghatika = $\frac{1}{60}$ day = 60 vighatsi.

§ 8. Series and Sequences

The problems on series and sequences are very original, varied and interesting. Kaye, however, holds a different opinion and gives a very confused and misleading analysis of the types of problems. In most of the problems, the sum to n terms happens to be proportional to the first term and therefore Ishtakarma can apply. The several types are indicated below in modern notation with T_n and S_n denoting respectively the nth term and the sum to n terms. We discuss *eight* types according to the character of T_n.

(1) *Arithmetical Progression.*

$T_n = nT_1 \pm (a + d \cdot \overline{n-1})$†. This can be reduced to

$T_n \pm d = n(T_1 \pm d) \pm a$, or changing the notation

$V_n = nV_1 + a$, which is an arithmetic progression with common difference V_1 and first term $V_1 + a$.

(2) *Geometrical Progression.*

$$T_n = ar^{n-1} + ab\,\frac{r^{n-1}-1}{r-1} \quad \dagger\dagger$$

$$= a\left\{\frac{b+r-1}{r-1}\cdot r^{n-1} - \frac{b}{r-1}\right\}$$ which is derived from a G.P. by increasing every term by a constant.

$$\therefore \quad S_n = a\left\{\frac{b+r-1}{(r-1)^2}\cdot(r^n-1) - \frac{nb}{r-1}\right\}$$ which is proportional to the first term a if b, r, n be given.

Illustration : $S_n = 329$, $r = 3$, $n = 5$, $b = \frac{3}{4}$. ††

Hence $a = 2$, obtained in the text by Ishtakarma assuming $a = 1$ and setting the corresponding S_n (from which the true value of a comes out by proportion).

(3) $T_n = a\ S_{n-1}$.*

Hence $S_n = S_{n-1} + T_n = S_{n-1}(1+a) = S_{n-2}(1+a)^2 = \cdot\ \cdot\ \cdot\ \cdot\ \cdot$
$= S_1(1+a)^{n-1}$ or $T_1(1+a)^{n-1}$

(4) $T_n = n \cdot S_{n-1}$**.

This is the most interesting of all.

---

† B. M. II P. 122. 22 verso.

†† B. M. II P. 143, 51 verso; B. M. III p. 164.

* B. M. III P. 236 (42, verso) where $S_n = 54$, $a = \frac{1}{2}$, $n = 3$. Hence $T_1 = 24$.

** B. M. III P. 194 (23, verso) where $S_n = 300$, $n = 4$.
Hence $T_1 = 5$, $T_2 = 10$, $T_3 = 45$, $T_4 = 240$,

Since $T_n = S_n - S_{n-1} = S_n \cdot \frac{n}{n+1}$, $S_{n-1} = \frac{S_n}{n+1}$.

Similarly $T_{n-1} = S_{n-1} \cdot \frac{n-1}{n} = \frac{n-1}{n(n+1)} S_n$ and $S_{n-2} = \frac{S_n}{(n+1)n}, \cdot \cdot \cdot$

$$T_{n-r} = \frac{(n-r)\, S_n}{(n+1)n \cdot \cdot \cdot (n-r+1)},$$

$$S_{n-r-1} = \frac{S_n}{(n+1)n \cdot \cdot \cdot \cdot (n-r+1)} \quad (0 < r < n-1)$$

$$T_2 = \frac{2S_n}{(n+1)n \cdot \cdot \cdot 3} = 2T_1.$$

Hence $T_1 = \frac{2S_n}{(n+1)\,!}$ and $S_n = \frac{1}{2} T_1 \cdot (n+1)\,!$

Thus, given the law of formation of the sequence, and the sum to n terms, and the number of terms n, the series can be determined. Here again the sum is proportional to the first term and 'Isḥṭa-karma' operation is applicable. This is the method adopted in the text which is indeed preferable in numerical work to the algebraic expressions that we have given.

(5) $T_n = nS_{n-1} \pm (a + n - 1 \cdot d) = nS_{n-1} + U_n$ (say).

$\therefore \quad S_n = (n+1)\, S_{n-1} + U_n$

$S_{n-1} = nS_{n-2} + U_{n-1}$

. . . .

$S_2 = 3S_1 + U_2 = 3T_1 + U_2$

Hence, eliminating $S_{n-1}, S_{n-2}, \cdot \cdot \cdot S_2$, we have

$$S_n = U_n + (n+1)\, U_{n-1} + (n+1) \cdot nU_{n-2} + \cdot \cdot \cdot + (n+1)\, n\, (n-1) \cdot \cdot \cdot 4U_2$$
$$+ 3 \cdot (n+1)\, n\, (n-1) \cdot \cdot 4 \cdot T_1$$
$$= \tfrac{1}{2} T_1 \cdot (n+1)\,! + U_n + (n+1)\, U_{n-1} + \cdot \cdot \cdot$$
$$+ (n+1)\, n\, (n-1) \cdot \cdot \cdot 4U_2$$

The example in the text † is somewhat like this:

A has something with $1\frac{1}{2}$, B twice as much as A with $2\frac{1}{2}$, C three times as much as A and B and $3\frac{1}{2}$ in addition, D four times as much as A, B, and C and $4\frac{1}{2}$ in addition, their total possessions are 222. What is the possession of A?

This is interpreted by Kaye symbolically as:

$A = x\ (1 + 1\frac{1}{2})$, $B = 2A + 2\frac{1}{2}x$, $C = 3A + 3B + 3\frac{1}{2}x$, $D = 4A + 4B + 4C + 4\frac{1}{2}x$, and $A + B + C + D = 222$.

† B. M. II 24 verso p. 124.

I believe this interpretation is incorrect though it may justify the solution of the text and the use of Ishtakarma. A proper interpretation would be

$$A = x + 1\tfrac{1}{2},\ B = 2A + 2\tfrac{1}{2},\ C = 3A + 3B + 3\tfrac{1}{2},\ D = 4A + 4B + 4C + 4\tfrac{1}{2}$$

$$A + B + C + D = 222.$$

Though $x = 1$ satisfies by chance this set of equations, the set of equations is not strictly amenable to Ishtakarma by mere proportion. A different kind of Ishtakarma similar to the one described on p. 7 above should have been adopted here with due care. The Bakshali mathematician has not done this and therefore has erred in his method. Such an error has been repeated in three similar problems †† and we get a glimpse of the limitations of the Ganakaraja.

(6) $T = {}_n T_{n-1}$

Here, $T_n = n!$ if $T_1 = 1$ and there is no compendious formula for the exact summation of such terms.

(7) $T_n = nT_{n-1} \pm (a + \overline{n-1} \cdot d)$

This can be reduced to the form $U_n = nU_{n-1} \pm a$, which also does not admit of summation by any formula.

(8) $T_n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})a_n.$

We are reminded of Euler's identity :

$$(1 - a_1) + a_1 (1 - a_2) + a_1 a_2 (1 - a_3) + \cdots + a_1 a_2 \cdots a_{n-1} (1 - a_n)$$
$$= 1 - a_1 a_2 \cdots a_n.$$

Changing a_r to $1 - a_r$, we have the Indian counterpart of Euler's identity, viz.

$$a_1 + a_2 (1 - a_1) + a_3 (1 - a_1)(1 - a_2) + \cdots + a_n (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})$$
$$= 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_n).$$

Problems involving this identity are common in Hindu mathematical works from at least the 8th century onwards. No wonder that the Arithmetical Papyrus of Akhim (9th century?) contains such problems. But Kaye would prefer to make Akhim the inventor of the above identity and hence infer that the Bakshālī Manuscript must belong to the tenth century or later.

†† B. M. II 24 recto, 25 recto and verso, pp. 123–125.

§ 9. General Remarks

Having pointed out the famous peculiarities of the Bakshālī Manuscript in respect of its mathematical content, exposition and symbolism we will conclude with a few general remarks on the age and style of the work. Peculiarities apart, the Bakshālī text is more or less a replica of other Hindu mathematical works, such as the Ganitasara Sangraha. It contains in common with them the following :

(1) Practical and commercial problems such as the computation of fineness of gold, (2) problems on income and expenditure, (3) motion-problems, (4) profit and loss, (5) interest, (6) bills of exchange or hundika, and (7) miscellaneous problems involving literary and social references.

In addition, we may note that the way in which the solutions are given in the Bakshālī Manuscript in such a general form as to be nearly algebraic in character even though no adequate symbolism is employed, is just characteristic of all Indian works, since the day when Aryabhata set the fashion in this respect. There are, however, certain omissions, for example, expressions for the sums of squares and cubes, the indeterminate equations of the first and second degree, shadow problems, permutation and combination. These may be either apparent or real, because our manuscript is mutilated and several leaves are stuck together. In the latter case, the omission is significant and bespeaks an early date of composition, i.e. prior to the tenth century by which period these problems had become well known at least in Indian works. Among further evidences of early composition, we have already mentioned the meagre elaboration of problems dealt with in greater detail in the works of Brahmagupta Mahaviracharya and Bhaskara, the studied absence of the word-numeral notation, the use of the cross-symbol † instead of the dot ⊙ to denote a negative quantity, the use of the dot ⊙ to denote an unknown quantity, and a certain careless application of the Ishṭa-karma in as many as five problems.

The employment of the modern place-value arithmetical notation is regarded by Kaye as indicating a late period, as also what he believes to be non-Indian material, viz. the negative sign resembling the Diophantine symbol, the use of the Regula-falsi, the square-root approximation, the use of the Sexagesimal notation for example in the conversion of days into ghatikas, vighatis, and so forth, and the use of the terms dinara, dramma, satera and so forth. How much the

non-Indian material in Bakshālī Manuscript is really indigenous has been sufficiently brought out in the preceding pages.

However late we may wish to place the manuscript, we cannot well go beyond the tenth century. We therefore come near the times of Sridhara, Mahavira, and Chaturveda with whose works the Bakshālī work has many points in common. Evidences of language, script and such special terms as Hundika do not contradict this view. Kaye's insistence of the twelfth century as the probable date of composition is wholly untenable.

The systematic presentation of the working steps and methods of verification, along with the use of a certain kind of syncopated notation are of great pedagogic interest, quite in keeping with the fact that the work was written for the express benefit of young boys†. The simple and diffuse style adopted suits this aim.

The Bakshālī Manuscript may lack the subtlety of the Aryabhatiya; it may not possess the profundity of the Brahmasphuta Siddhanta; it may be a poor literary specimen by the side of the rich poetry of Ganitasarasangraha; but no one can deny its great value for a practical teacher. If we recollect that Bakshālī was at a distance of only 70 miles from Taxila, one of the renowned University centres in Ancient India, we have here perhaps a glimpse of the lecture-notes of a University professor of bygone days. To do the Ganakaraja justice, we may add that flashes of genius gleam forth here and there in the method of reconciliation of square-root approximations, in dreaming of series and sequences governed by elegant and sometimes complicated laws, in the solution of systems of linear equations by suitable change of variables and assumption of tentative solutions, and lastly in the construction of problems of humorous and mythological interest.

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† B. M. II p. 142, 50*r*. The Colophon reads:
. . . . Vasishta putraha.
Sikasyartheputra pautra upayogam bhavatu likhi.
tam chchhajaka putra ganaka rāja brahmanena . . .

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TOPICS IN THE THEORY OF ANALYTIC FUNCTIONS (II):*

(Singularities, Over-convergence and Gaps; Harmonic Functions; Maximum-modulus principle; Normal Families of Analytic Functions.)

BY

V. GANAPATHY IYER

1. **Singularities of analytic Functions.** An analytic Function (according to Weierstrass, I, § 7 – 10) is entirely determined from any one of its elements. Therefore, theoretically, the singularitios of an analytic function must be completely determinable if we are given a single element of the function. But the practical determination of the position and nature of the singularities of a function starting from one of its elements is extremely difficult. We have seen that there is at least one singular point on the circumference of the circle of convergence of a power series. Since an analytic function is merely the totality of its elements it follows that a singular point of an analytic function is either (i) situated on the circumference of the circles of convergence of its elements or (ii) a limit point of the singularities of type (i). So it is sufficient to study the nature and position of the singularities on the circumference of the circles of convergence of a power series.

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* The first article appeared in *The Mathematics Student*, Vol. 5, (1938), pp. 1 – 24. This will be referred to as I in the sequel. The notations in I are used throughout.

1·1. *The general test for a singularity*†.

Let $P_a(z)$ be the power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n \qquad \dots \quad (1)$$

Let z_0 be a point on the circumference c (P_a) of the circle of convergence C (P_a) of the series. *Then z_0 is a singular point if and only if the radius of convergence of the immediate continuation P_b (z) of P_a (z) for any point b on the radius (a, z_0) is exactly equal to $|b-z_0|$.* The condition we arrive at by using the above criterion is, in general, unwieldy and so is not useful in practice. Simplifications* of the test have been introduced and with their aid several interesting results on the position of singularities have been obtained*. We shall confine ourselves to mentioning a few of these.

1·2. In stating the results we shall take $a=0$ in P_a (z) so that we get

$$P_0(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \dots \quad (2)$$

If $a_n \geqslant 0$, or more generally, $|$ amp. $(a_n)| \leqslant \alpha < \pi/2$ *then $z=\rho$ on $c(P_0)$ is a singular point where ρ is the radius of convergence.* Again, let $[\lambda_n]$, $0 \leqslant \lambda_1 < \lambda_2 \cdots < \lambda_n \cdots$ denote the indices of (2) for which $a_{\lambda_n} \neq 0$, that is, the coefficients are non-vanishing. Then, if $\lambda_n/n \to \infty$ *as* $n\to\infty$, *every point on c (P_0) is a singular point. In particular this is true if (i) $\lambda_{n+1}-\lambda_n \to \infty$ and (ii) $\lim \lambda_{n+1}/\lambda_n > 1$.* In this case the analytic function cannot be continued beyond the circle C (P_0) which is therefore its domain of existence. The function is uniform in C (P_0) and the circumference c (P_0) is called the natural cut or boundary for the function. In this connection we wish to mention the following curious result. Let us start with *any* series $\sum a_n z^n$ and let us consider the totality of the series of the form $\sum \pm a_n z^n$ obtained by distributing in all possible manner the signs + and − among the coefficients (a_n) of the given series. Then we can find at least one among these series for which c (P_0) is the natural boundary. Indeed, the cardinal number of such series is the same as that of the continuum.**

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† See P. Dienes, *The Taylor Series*, Oxford (1931), the chapter on singularities.

* See P. Dienes, last cited. Also, S. Mandelbrojt, "Modern Researches on the Singularities of functions defined by Taylor's Series", *The Rice Institute Pamphlet*, Vol. 14 (1927), No. 4.

** For proofs of these and further results see P. Dienes and Mandelbrojt last cited.

2. Uniform convergence in a domain. Let $[f_n(z)]$ be a sequence of functions each of which is regular in a domain D. *If the sequence converges uniformly on any closed set interior to D then the sequence is said to converge uniformly in D.* In this case the limit function is also regular in D; here we suppose that a constant (finite or infinite) is regular in D, the limit of the sequence being an infinite constant when $|f_n(z)| \to \infty$ uniformly on any closed set interior to D. In this connection we wish to mention Weierstrass's theorem on the uniform convergence of a sequence of functions. Let D be a domain such as contemplated in Cauchy's formula [(10), § 3·6, p·8, I]. Let $[f_n(z)]$ be a sequence of functions each of which is regular in D and continuous on the boundary B (D). Then the uniform convergence of $[f_n(z)]$ on B (D) implies its uniform convergence in D. Moreover, the sequence $\left[f_n^{(k)}(z)\right]$ of the kth derivatives also converges uniformly in D to $f^{(k)}(z)$ where $f(z)$ (here supposed to be finite) is the limit of the original sequence.

2·1. **Over-convergence.** Let us consider a sequence $[f_n(z)]$ of functions converging uniformly in a domain D and let us further suppose that the sequence $[f_n(z)]$ does *not* converge uniformly in any *larger* domain containing D. If this be the case we might call D a *complete domain** of uniform convergence of the sequence $[f_n(z)]$. Now although the whole sequence $[f_n(z)]$ does not converge uniformly in a larger domain D_1 containing D it *can happen* that a *sub-sequence* $[f_{n_k}(z)]$ of $[f_n(z)]$ *converges uniformly in such a domain* D_1. When this is the case *the original sequence is said to over-converge outside D.* In this case the limit function $f(z)$ of $[f_n(z)]$ is necessarily regular in the larger domain D_1 and the sub-sequence converges to $f(z)$ in D_1 as well. In order that a sequence over-converges outside D it is necessary and sufficient that a sub-sequence of $[f_n(z)]$ should converge uniformly in the neighbourhood of a point of B(D).

2·2. The phenomenon of over-convergence explained in the previous paragraph has been mostly studied in the case when $f_n(z) = S_n(z) = \sum_{k=0}^{n} a_k z^k$, the sum to first $(n+1)$ terms of the power series $P_0(z)$. By the well-known properties of a power-series, $[S_n(z)]$ cannot converge at any point outside C (P_0). So the interior of C (P_0) is a complete domain of uniform convergence of $[S_n(z)]$.

* A sequence can converge uniformly to different functions in different non-overlapping domains. So there can be more than one complete domain of uniform convergence, see I, § 6.

The main result in connection with the over-convergence of the sections $[S_n(z)]$ of a power-series $P_0(z)$ is due to A. Ostrowski*. *Let* $P_0(z)$ *be such that there are an infinity of pairs of integers* (p_k, q_k), $k=1, 2, \ldots$ *such that* (*i*) $a_n=0$ *for* $p_k < n < q_k$ *and* (*ii*) $\lim_{k\to\infty} \frac{q_k}{p_k} > 1$. *Then the sub-sequence* $[S_{p_k}(z)]$ *of* $[S_n(z)]$ *converges uniformly in the neighbourhood of any regular point* z_0 *of the function* $f(z)$ *defined by* $P_0(z)$, z_0 *being on* $c(P_0)$. That is, under these conditions $[S_n(z)]$ over-converges outside $C(P_0)$.

2·3. **Gaps.** In the above result on over-convergence we come across *an infinity* of *gaps* in the coefficients $(a_0, a_1, a_2, \ldots)$ of the power series $P_0(z)$. These gaps (p_k, q_k) are such that the width $q_k - p_k$ of the gaps are not less than dp_k $(d>0$ fixed) for all $k \geqslant$ some k_0. When a power series $P_0(z)$ possesses such gaps we shall call it a *gap-series* (*seriés lacunaires*). The hypothesis on the existence of gaps in connection with the theorem on over-converges is not accidental. It can be proved that if $[S_n(z)]$ over-converges then

$$P_0(z) = Q_0(z) + R_0(z) \qquad \ldots \quad (3)$$

where $Q_0(z)$ is a gap-series and $R_0(z)$ has a greater radius of convergence than $P_0(z)$. The over-convergence theorem in § 2·2 easily leads to the case (ii) of the second result on singularities mentioned in § 1·2.†

3. Harmonic functions. A real function $u(z) = u(x, y)$, $z = x + iy$. is said to be harmonic in a domain D when it is the real part of a function $f(z) = u(z) + iv(z)$ regular in D. Since $i\,f(z)$ is also regular in D, the complex or the imaginary part $v(z)$ is also harmonic in D. From (6) of I (p. 6) we deduce that any harmonic function $u(z)$ is a solution of the Laplace's equation

$$\triangle U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \qquad \ldots \quad (4)$$

Conversely any continuous solution of (4) is a harmonic function.

3·1. Let $z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$ and let $u(z) = u(x, y) = u(r, \theta)$ be a function harmonic in a domain D. Let z_0 be any point of D

* See P. Dienes, last cited, the chapter on Over-convergence and Gap theorems See also E. C. Titchmarsh, *Theory of Functions*, pp. 220-224.

† For proofs of these results see P. Dienes, last cited. Also, see Bourion *L'ultraconvergence dans les series de Taylor*: Actualites Scientifiques No. 472.

and let C (z_0, r) be a circle in D. Then one of the most important properties of harmonic functions is given by the relation

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta \qquad \ldots (5)$$

That is, *the mean value of* $u(z)$ *along the circumference of the circle* $C(z_0, r)$ *is equal to its value at the centre.* This result is easily proved by integrating a power series along the circumference of a circle and equating the real parts. Conversely *any function* $u(z)$ *continuous in a domain* D *satisfying the relation* (5) *for any circle inside* D *is harmonic in* D.

3·2. **Sub-harmonic and Super-harmonic functions.** Let $u(z)$ be a real function continuous in a domain D. If *for every circle* $C(z_0, r)$ inside D the relation

$$u(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta \qquad \ldots (6)$$

holds, the function $u(z)$ is said to be *sub-harmonic* in D. If, on the other hand, the relation

$$u(z_0) \geqslant \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta, \qquad \ldots (7)$$

holds, $u(z)$ is said to be *super-harmonic* in D. If $u(z)$ is sub-harmonic in D, then $-u(z)$ is super-harmonic in D. *A harmonic function is at the same time sub-harmonic and super-harmonic.*

3·3. One of the chief properties of a sub-harmonic function $u(z)$ defined in a domain D is that $u(z)$ *cannot attain its upper bound (or maximum value) at a point of* D *unless* $u(z)$ *is a constant in* D. This result is easily proved from relation (6). If $u(z)$ is continuous on the boundary B (D) this upper bound will be attained at a point of B (D). Similarly *a super-harmonic function cannot attain its lower-bound (or minimum value) at a point of* D. Therefore *a function* $u(z)$ *harmonic in* D *cannot attain its upper or lower bound (maximum or minimum value) at a point of* D *unless it is a constant.* From this we deduce that *if* $u_1(z)$ *and* $u_2(z)$ *are harmonic in* D *and continuous on* $B(D)$ *then the equality* $u_1(z) = u_2(z)$ *at points of* $B(D)$ *implies* $u_1(z) \equiv u_2(z)$ *throughout* $D + B(D)$.

3·4. **Dirichlet's problem.** The most extensively studied subject in connection with the theory of harmonic functions is what is known as Dirichlet's problem. The last statement in § 3·3 above shows that

if u (z) is harmonic in a domain D and continuous on B (D) then there cannot exist another function harmonic in D taking the same set of continuous boundary values on B (D). Dirichlet's problem concerns itself with the solution of the problem arising from the converse of this proposition. Let $v(z)$ be a *given* function continuous on the boundary B (D) of a domain D. Is it possible to find a function $u(z)$ harmonic in D such that the boundary value of $u(z)$ on B (D) is just $v(z)$? With suitable restrictions on the nature of the boundary B (D) of the domain D the above problem has been completely solved. References are made elsewhere* for a detailed account of these and further properties of harmonic functions.

4. The maximum-modulus principle for analytic functions. Let $f(z)$ be a function regular in a domain D. Then $|f(z)|$ *cannot attain its upper bound* (*or maximum*) *at a point of D unless $f(z)$ is a constant.* Though this result can be proved directly it is a special case of the results of § 3·3 since it can be shown that $|f(z)|$ is sub-harmonic in D. The above result is known as the maximum-modulus principle for analytic functions. If $|f(z)|$ is continuous on B (D) then the upper bound is attained at a point of B (D). If $f(z) \neq 0$ in D, then $\frac{1}{f(z)}$ is regular in D and then we can also conclude that $|f(z)|$ cannot attain its minimum at a point of D,

4·1. The maximum-modulus principle has extensive applications in the theory of analytic functions. It can be extended to mutli-form functions if $|f(z)|$ is single-valued in the domain considered. As simple applications of this principle we can deduce that if $f(z)$ be regular in a circle $|z| < R$ and $M(r, f) = \max_{|z| \leqslant r} |f(z)|, \; 0 \leqslant r < R$, then $M(r, f)$ is an increasing function r and $\log M(r, f)$ is a convex function† of $\log r$. The last statement includes what is known as Hadamard's three circles theorem. Numerous other interesting results could be deduced from this principle.‡

5. Normal families of analytic functions. A class or a family $\{f(z)\}$ of functions each defined and regular in a domain D is said to be *normal in D when any sequence $[f_n(z)]$ of functions selected from the family contains a sub-sequence converging uniformly in D* (§ 2 above).

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* See E. Goursat, *Cours d' Analyse Mathematique*, Tome III.

† A function $\phi(x)$ is said to be convex in x for $a \leqslant x \leqslant b$ if

$$\phi(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 \phi(x_1) + \lambda_2 \phi(x_2)$$

where x_1 and x_2 are any two points in (a, b) and λ_1 and λ_2 are numbers such that $\lambda_1 + \lambda_2 = 1$

‡ See P. Dienes, and E. C. Tithmarsh, last cited.

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The idea here closely resembles the well-known result that *any infinite set* (*or class*) of numbers (real or complex) has *at least one limit point*, that is, any sequence selected from the set contains a sub-sequence converging to a limit (finite or infinite). Often, in higher analysis,* we have to consider classes of elements for which the notions of convergence and limit could be defined. But it is *not necessary* that any infinite set of these elements should possess the property mentioned above for numbers. In such cases it becomes important to consider classes of such elements which possess this special property. This is the idea behind the notion of a normal family. If we consider a class or family of functions regular in a domain D and if "*ordinary*" convergence (in the case of numbers) be replaced by "*uniform convergence in D*", then the family is normal when it has the property specified above for numbers.

5·1. We shall say that a family $\{f(z)\}$ of functions regular in a domain D is *essentially bounded* in D if every point z_0 of D is the centre of a circle $C(z_0, r)$ lying in D such that

$$|f(z)| \leqslant M, \text{ for } z \text{ in } C(z_0, r), \qquad \ldots \quad (8)$$

and for each function $f(z)$ of the family; here M is a finite number which may vary with z_0 and r but is independent of the particular function $f(z)$ of the family. The most important test for a normal family can be stated as follows. *A family $\{f(z)\}$ of functions regular in D is normal in D if the family is essentially bounded in D.*

5·2. **Exceptional values and normal families.** There is an intimate connection between the values which the functions of a family omit to take and the property of the family being normal. *Let each function $f(z)$ of a family $\{f(z)\}$ of functions regular in a domain D omit to take two specified values α and β, $\alpha \neq \beta$. Then the family is normal in D* †

5·3. **Uniform convergence and normal families.** It would take us too far to consider exhaustively the results and applications of the theory of normal families. We shall content ourselves with a few general remarks. Firstly, it is possible to considerably generalise the theorem of Weierstrass on uniform convergence (§ 2) when we consider normal families. *Let $[f_n(z)]$ be normal in a domain D. If $[f_n(z)]$ converges at a set of points in D having at least one limit point in D, then the sequence converges uniformly in D.*

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* See, for instance, S. Banach, *Operations Lineaires*, Chaps. 1—4.

† The classical work in connection with the theory of normal families is P. Montel's "*Leçons sur les familles normales des fonctions analytiques*" (Borel Tracts).

5·4. **The limit functions of a normal family.** We have pointed out in § 2 that when a sequence of functions regular in D converge uniformly in a domain the limit function is either regular (in the ordinary sense in D) or is a constant finite or infinite*. In connection with normal families we can agree to regard a finite or infinite constant as regular in any domain D. *Then we can state that the class of limit functions of the various sub-sequences selected from a normal family consists of regular functions.*

5·5. Let $\{f(z)\}$ be a family of functions *regular and normal* in a domain D. *Then if the family be bounded at a single point* z_0 *of* D, *then the family is essentially bounded in* D. *The same conclusion is true if the class of limit functions does not include the infinite constant.* Again, *if a family* $\{f(z)\}$ *is normal in* D, *the number of roots of* $f(z)-a=0$ *in any closed region* R *in* D *is bounded as* $f(z)$ *varies over the functions of the family unless the constant* a *is included in the class of limit functions of the family.* These are some of the interesting results derived from the theory of normal families.†

5·6. **Schottky's theorem.** Let $f(z)$ be regular in the circle $|z|<R$. Let $f(z)$ omit to take the two values 0 and 1 in the circle. Let $|f(0)|\leqslant\delta$ and $|1-f(0)|\geqslant\delta$.

$$\textit{Then} \qquad |f(z)|\leqslant M(r,\delta),\ |z|\leqslant r<R \qquad \dots (9)$$

where $M(r,\delta)$ *depends only on* r *and* δ *and not on the particular function satisfying the conditions.* This result is known as Schottky's theorem. It has an intimate connection with the theory of normal families. In fact, Schottky's theorem along with the results of § 5·1 can be used to prove the test of exceptional values given in § 5·2. Conversely, if the test of 5·2 be assumed, Schottky's theorem‡ could be derived by using the properties mentioned in § 5·5.

5·7. In conclusion we wish to mention that the definition and properties of normal families can be extended to a family of functions *meromorphic* in a domain D (that is, having only poles in D). The theory of normal families have important and extensive applications in the theory of integral and meromorphic functions and in the subject of conformal representation.** In this connection it may be

* See § 2 for the definition of an infinite limit function.

† For these and other results see P. Montel, last cited. Also, G. Valiron, *Sur les valeurs exceptionnelles des fonctions méromorphes et de leurs dérivées* (Actualites Scientifiques et Industrielles, No. 570, 1937.)

‡ See P. Dienes and E. C. Titchmarsh last cited.

** See C. Caratheodory, "*Conformal Representation*" (Cambridge Tracts),

pointed out that the proof * of the test of exceptional values for normal families given in § 5·2 is essentially based on the topological fact that a *simply connected domain having at least two boundary points can be conformally represented on a finite circle.*

* See P. Montel, last cited.

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4. P. Dienes, *The Taylor Series*

5. J. Hadamard, *La Serié de Taylor et ses prolongement analytique.*

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9. P. Montel, *Leçons sur les familles normales des fonction analytiques.*

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Further references will be found in (4) and (10).

SOME DIOPHANTINE PROBLEMS AND NUMERICAL IDENTITIES

BY

ALFRED MÖESSNER, *Nurnberg, N. Germany*

1. I give firstly a solution of

$$A_1, A_2, A_3, A_4, A_5, A_6, \overset{4}{=} B_1, B_2, B_3, B_4, B_5, B_6 \qquad \ldots \quad (1{\cdot}1)$$

by which is meant that the sum of the nth powers of the numbers A_r is equal to the sum of the nth powers of the numbers B_r for $n=1, 2, 3,$ and 4.

THEOREM. *Given that*

$$\left.\begin{aligned} & x_1+x_2+x_3=y_1+y_2+y_3=s \\ \text{and} \quad & x_1{\cdot}x_2{\cdot}x_3=y_1{\cdot}y_2{\cdot}y_3 \end{aligned}\right\} \qquad \ldots \quad (1{\cdot}2)$$

then we have also

$$x_1, x_2, x_3, s-y_1, s-y_2, s-y_3 \overset{4}{=} y_1, y_2, y_3, s-x_1\ s-x_2, s-x_3$$

A solution for the equations (1·2) is given by

$$x_1=(ab-cd)\,a \qquad y_1=(ab-cd)\,c$$
$$x_2=(ab-cd)\,b \qquad y_2=(ab-cd)\,d$$
$$x_3=(a+b-c-d)\,cd \qquad y_3=(a+b-c-d)\,ab$$

Example: Take $a=b=3$, $c=4$, $d=1$ and we have $15+15+4=20+5+9$; $15{\cdot}15{\cdot}4=20{\cdot}5{\cdot}9$; $s=34$ and we get

$$15, 15, 4, 14, 29, 25 \overset{4}{=} 20, 5, 9, 19, 19, 30$$

2. Under certain circumstances we get solutions of (1·1) with only 5 numbers on each side. Thus $1+9+10=2+3+15$ and $1{\cdot}9{\cdot}10=2{\cdot}3{\cdot}15$. Here $s=20$ and the above method gives

$$1, 9, 10, 18, 17, 5 \overset{4}{=} 2, 3, 10, 11, 15, 19$$

from which 10 may be struck off on both sides.

3. Let us further impose the condition that A_1 A_2 A_3 and B_1 B_2 B_3 are to be perfect squares, and write $A_r=M_r^2$, $B_r=P_r^2$, $(r=1, 2, 3)$.

By solving the equations †

$$M_1^2 + M_2^2 + M_3^2 = P_1^2 + P_2^2 + P_3^2$$
$$M_1^2 \cdot M_2^2 \cdot M_3^2 = P_1^2 \cdot P_2^2 \cdot P_3^2$$

we obtain the required solutions.

Example: $69^2 + 18^2 + 19^2 = 57^2 + 46^2 + 9^2 = 5446$

and $69^2 \cdot 18^2 \cdot 19^2 = 57^2 \cdot 46^2 \cdot 9^2$

whence we have

$$69^2, 18^2, 19^2, 2197, 3330, 5365 \overset{4}{=} 57^2, 46^2, 9^2, 685, 5122, 5085$$

4. To obtain solutions of (1·1) where A_1, A_2, A_3, B_1, B_2, B_3 are perfect cubes, we write

$$A_1 = U_1^3,\ A_2 = U_2^3,\ A_3 = U_3^3;\ B_1 = V_1^3,\ B_2 = V_2^3,\ B_3 = V_3^3$$

where

$$U_1 = x\ (1 + 2x^3),\ U_2 = -x^2\ (2 + x^3),\ U_3 = 1 - x^3$$
$$V_1 = 1 + 2x^3,\ V_2 = x\ (2 + x^3),\ V_3 = x^2\ (1 - x^3)$$

and x is arbitrary.

Example: Put $x = 2$. We thus obtain

$$(-34)^3, 40^3, 7^3, 29952, 17039, 3087 \overset{4}{=} (-17)^3, 20^3, 28^3, -14305, 89039, 24696$$

5. To obtain solutions of (1·1) where A_1, A_2, A_3, B_1, B_2, B_3 are perfect 4th powers, we write

$$A_1 = W_1^4,\ A_2 = W_2^4,\ A_3 = W_3^4;\ B_1 = Z_1^4,\ B_2 = Z_2^4,\ B_3 = Z_3^4$$

We have now to obtain solutions of

$$W_1^4 + W_2^4 + W_3^4 = Z_1^4 + Z_2^4 + Z_3^4$$
$$W_1 \cdot W_2 \cdot W_3 = Z_1 \cdot Z_2 \cdot Z_3 \qquad \ldots \quad (5\cdot1)$$

After we have obtained a solution of the equation $C^4 + D^4 = E^4 + F^4$ with the help of Euler's formula, we write

$$W_1 = F^2,\ W_2 = CE,\ W_3 = DE;\ Z_1 = E^2,\ Z_2 = CF,\ Z_3 = DF$$

which will be found to satisfy (5·1).

For example with $C = 134$, $D = 133$, $E = 158$, $F = 59$ we have $s = W_1^4 + W_2^4 + W_3^4 = 3481^4 + 21172^4 + 21014^4$. The solution of (1·1) is now $A_1 = 3481^4$, $A_2 = 21172^4$, $A_3 = 21014^4$, $A_4 = (s - 24964)^4$
$A_5 = (s - 7906^4)$, $A_6 = (s - 7847^4)$; $B_1 = 24694^4$
$B_2 = 7906^4$, $B_3 = 7847^4$, $B_4 = (21172^4 + 21014^4)$
$B_5 = 3481^4 + 21014^4$, $B_6 = 3481^4 + 21172^4$.

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† Vide: Möessner and Gloden: Diophantische Probleme *The Proc. of the Indian Academy of Sciences VIII, p. 203.*

Since the system of equations $G_1^n + G_2^n + G_3^n = H_1^n + H_2^n + H_3^n$; $G_1^n G_2^n G_3^n = H_1^n H_2^n H_3^n$ for $n > 4$ has not been solved yet, we cannot get general solutions of (1·1) where $A_1 A_2 A_3 B_1 B_2 B_3$ are perfect fifth or higher powers.

6. Here is an elementary solution in integers of the equation

$$x + y = u^2; \quad x^3 + y^3 = t^2$$

We write

$$x = (a^2 + 3b^2 - 2ab)^2 - (a^2 - b^2)^2$$
$$y = (a^2 - b^2)(a^2 - 4ab + 7b^2)$$
$$u = a^2 - 4ab + b^2$$
$$t = (a^4 - 2a^3b + 6a^2b^2 - 14ab^3 + 13b^4)(a^2 - 4ab + b^2).$$

Example: Put $a = 3$, $b = 2$ and we have

$$56 + 65 = 11^2$$
$$56^3 + 65^3 = 671^2.$$

7. Here are some new solutions of Tarry's problem:

1, 4, 6, 8, 15, 19, 20, 20, 22, 27, 29, 34, 38, 41, 42, 57, 58, 61, 65, 70, 72, 77, 79, 79, 80, 84, 91, 93, 95, 98 $\stackrel{4}{=}$ 2, 2, 7, 9, 14, 16, 21, 21, 25, 28, 28, 30, 40, 40, 44, 55, 59, 59, 69, 71, 71, 74, 78, 78, 83, 85, 90, 92, 97, 97.

By using theorems on simultaneous identities already proved, we obtain from the above the relation

$(2, 6, 6, 11, 15, 16, 18, 18, 21, 30, 33, 33, 34, 44, 47, 49, 52)^n$

$= (4, 5, 8, 9, 13, 13, 17, 23, 26, 27, 28, 32, 40, 42, 46, 51, 51)^n$ for $n = 1$ 3, 5, 7, 9, 11, 13.

On the Normalizing Factors for Legendre Polynomials

In all the treatments of Legendre polynomials known to me, the derivation of the relation

$$(1)\quad \int_{-1}^{1} P_n^2(u)\,du = \frac{2}{2n+1}$$

is fairly complicated.* I propose to give two simple proofs of (1) one based on each of the standard recursion formulas†

$$(2)\quad (n+1)\,P_{n+1}(x) - (2n+1)\,xP_n(x) + n\,P_{n-1}(x) = 0,$$

$$(3)\quad nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Proof from (2). We use, beyond (2), only the facts that $P_n(x)$ is of degree n, and orthogonal to every polynomial of degree $<n$.

Multiply (2) by $P_{n-1}(x)$ and integrate over $(-1, 1)$. We find

$$(4)\quad (2n+1)\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = n\int_{-1}^{1} P_{n-1}^2(x)\,dx.$$

In (2), replace n by $n-1$:

$$nP_n(x) - (2n-1)\,xP_{n-1}(x) + (n-1)\,P_{n-2}(x) = 0;$$

multiply by $P_n(x)$, and integrate, obtaining

$$(5)\quad n\int_{-1}^{1} P_n^2(x)\,dx = (2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx.$$

Comparing (4) and (5), we have

$$(6)\quad \int_{-1}^{1} P_n^2(x)\,dx = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)\,dx;$$

and since $P_0(x) \equiv 1$, $\int_{-1}^{1} P_0^2(x)\,dx = 2$, successive applications of (6) give (1).

Proof from (3). Here we use also the fact that $|P_n(\pm 1)| = 1$.

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* See, for example, O. D. Kellogg, *Foundations of potential theory*, 1929, p. 132; M. H. Stone, *Developments in Legendre polynomials*, Annals of Math. (2), vol. 27 (1925) pp. 315-329.

† Kellogg, p. 126, (7), and p. 127, (9); Stone, p. 316, (1) and (3).

Multiply (3) by $P_n(x)$ and integrate, obtaining since $P_n(x)$ is orthogonal to $P'_{n-1}(x)$,

$$n \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} xP_n(x)\,P_n'(x)\,dx$$

$$= \Big[xP_n^2(x)\Big]_{-1}^{1} - \int_{-1}^{1} P_n(x)\,[xP_n'(x) + P_n(x)]\,dx$$

$$= 2 - n\int_{-1}^{1} P_n^2(x)\,dx - \int_{-1}^{1} P_n^2(x)\,dx,$$

and hence (1).

The proof based on (2) can be extended to general orthogonal polynomials $\phi_n(x)$, which satisfy the recursion relation*

$$\phi_{n+1}(x) - (x - c_{n+1})\,\phi_n(x) + \lambda_{n+1}\,\phi_{n-1}(x) = 0 \quad (\phi_o = 1,\ \phi_{-1} = 0\,;\ \lambda_n > 0).$$

We obtain, if (a, b) is the interval of orthogonality of the $\phi_n(x)$, and $\psi(x)$ is the associated weight function,

$$(7) \quad \int_a^b \phi_n^2(x)\,d\psi(x) = [\psi(b) - \psi(a)]\,\lambda_{n+1}\,\lambda_n \cdots \lambda_1.$$

Relation (7) is well known in another form.†

Ong's Hat
New Jersey, U. S. A.
E. S. PONDICZERY.

A Note on some Diophantine Equations

1. Consider the Diophantine equations:

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2, \quad \ldots (1)$$

and

$$x^4 + y^4 + z^4 = u^4 + v^4 + w^4, \quad \ldots (2)$$

where

$$x = y + z \text{ and } u = v + w. \quad \ldots (3)$$

Substituting $y + z$ for x and $v + w$ for u in (1) and (2), we get

$$y^2 + yz + z^2 = v^2 + vw + w^2, \quad \ldots (4)$$

and

$$(y^2 + yz + z^2)^2 = (v^2 + vw + w^2)^2. \quad \ldots (5)$$

We are thus left to solve the equation (4) only.

Let $y - z = m$ and $v - w = n$;

so that

$$y = \tfrac{1}{2}(x + m),\ z = \tfrac{1}{2}(x - m)\,;\ v = \tfrac{1}{2}(u + n), \text{ and } w = \tfrac{1}{2}(u - n).$$

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* See, e.g., J. Shohat, *Théorie gènèrale des polynomes orthogonoaux de Tchebichef*, Mémorial des Sci. Math., fasc. 66 (1934), p. 24.

† Shohat, p. 25.

Substituting these values in (4), we now obtain

$$3x^2 + m^2 = 3u^2 + n^2,$$

or $$3(x+u)(x-u) = (n+m)(n-m) \; ; \qquad \ldots \quad (6)$$

which is an equation of the form[1]:

$$3rs = pq \cdot \qquad \ldots \quad (7)$$

As a solution of (7) we have

$$r = ab, \; s = cd \; ; \qquad p = 3ac, \; q = bd.$$

Hence we easily obtain[2]

$$x = \tfrac{1}{2}(ab+cd), \qquad u = \tfrac{1}{2}(ab-cd),$$
$$y = \tfrac{1}{4}(ab+cd+3ac-bd), \qquad v = \tfrac{1}{4}(ab-cd+3ac+bd),$$
$$z = \tfrac{1}{4}(ab+cd-3ac+bd) \; ; \qquad w = \tfrac{1}{4}(ab-cd-3ac-bd).$$

These provide rational solutions of (1), (2) and (3), in terms of four parameters: a, b, c and d. Integral solutions are easily deduced now.

Example. Taking $a = 13$, $b = 11$, $c = 7$ and $d = 5$,

we have $x = 89$, $y = 99$, $z = -10$; $u = 54$, $v = 109$, $w = -55$.

Hence $99^h + 89^h + 10^h = 109^h + 55^h + 54^h$, $h = 2, 4$.

Taking $ab = -cd$, it can be easily shown that the equations:

$$2\; x^2 = u^2 + v^2 + w^2,$$
$$2\; x^4 = u^4 + v^4 + w^4 \; ;$$

have solutions in integers.

As an example, we have $x = 7$, $u = 8$, $v = 5$, and $w = 3$.

2. We next consider the equations[3]:

$$3\; x^2 = u^2 + v^2 + w^2, \qquad \ldots \quad (8)$$
$$3\; x^4 = u^4 + v^4 + w^4. \qquad \ldots \quad (9)$$

Multiplying (9) by 3 and subtracting the square of (8), we obtain $0 = (u^2 - v^2)^2 + (v^2 - w^2)^2 + (w^2 - u^2)^2$.

Hence $u = v = w = x$.

Thus the equations (8) and (9) have no non-trivial solution.

References

[1] Such equations are considered in my paper entitled "A Problem in Diophantine Analysis." See the *American Journal of Mathematics*, Vol. 56, (1934), pp. 269—74.

[2] For another solution see *Jahresbericht*, 49, 1939, 9—11.

[3] Suggested by Alfred Moessner.

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HANSRAJ GUPTA.

ASTRONOMICAL NOTES

Planet Notes from June to August 1939

Note.—All times hereunder used are I. S. T. and the co-ordinates of planets given are for 5 hours 30 minutes I. S. T.

Sun. During the period the Sun moves from Taurus to Leo with increasing North declination and attains its maximum North Declination on 22nd June the date of Summer Solstice which marks the Summer season. After this the Sun moves with decreasing North Declination. On 5th July the earth is in aphelion.

The Sun's position on the 1st of each month will be as follows:—

		June	*July*	*August*
R. A.	...	4 h 32 m	6 h 36 m	8 h 41 m
Decl.	...	21° 54′ N	23° 11′ N	18° 19′ N

Moon. The following phases of the Moon will occur.

	Full Moon	*Last Quarter*	*New Moon*	*1st Quarter*
June	2nd at 8-41 A.M.	10th at 9-37 A.M.	17th at 7-7 P.M.	24th at 10-5 A.M.
July	1st at 9-46 P.M.; 31st at 0-7 P.M.	10th at 1-19 A.M.	17th at 2-33 A.M.	23rd at 5-54 P.M.
August	30th at 3-39 A.M.	8th at 2-48 P.M.	15th at 9-23 A.M.	22nd at 2-51 A.M.

The Moon will be nearest the earth on 20th June, 18th July and 15th August and will be away on 8th June, 5th July and 29th August.

Positions of the Moon.

		1st June	*1st July*	*1st August*
R. A.	...	15 h 37 m	18 h 2 m	21 h 7 m
Decl.	...	17° 43′ S	19° 21′ S	11° 26′ S

Mercury. Early in June and towards the latter part of August the planet is poorly placed for observation as a morning object, but during July it is favourably seen as an evening object. Mercury is stationary on two occasions viz., 26th July and 20th August. It is at its greatest elongation East on July 13th and West on 28th August. The planet is in superior conjunction with the Sun on 7th June and in inferior conjunction on 10th August. It is also in conjunction with the Moon on the 18th July.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	3 h 59 m	8 h 15 m	9 h 37 m
Decl.	...	20° 13′ N	21° 22′ N	9° 37′ N

Venus. The course of the planet during the period is from Taurus to Leo and its motion direct. Throughout the period Venus is a morning object and by the end of August it nears the Sun and is hardly visible. The planet is in conjunction with Uranus on 5th June, with Moon on the 15th June and with Regulus on the 25th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	2 h 47 m	5 h 17 m	8 h 1 m
Decl.	...	14° 29′ N	22° 31′ N	21° 13′ N

Mars. Mars is in Capricornus throughout. During the early part of June the planet rises by 9 P.M. and it rises earlier day by day so that by the end of August it is seen till 3 A.M. On June 24th Mars is stationary and its motion retrograde and again it is stationary on August 24th when its motion is changed direct. It is in opposition with the Sun on July 23rd and is in conjunction with the Moon on 6th June, 3rd, 30th July and 26th August.

		1st June	*1st July*	*1st August*
R. A.	...	20 h 19 m	20 h 32 m	20 h 3 m
Decl.	...	22° 32′ S	24° 5′ S	27° 3′ S

Jupiter. This planet remains in the constellation Pisces. Early in June Jupiter rises after midnight and by the end of August it rises by 8 P.M. On July 30th the planet is stationary and then it moves with retrograde motion. This planet is in conjunction with the Moon on 13th June, 11th July and 7th August.

		1st June	*1st July*	*1st August*
R. A.	...	0 h 15 m	0 h 29 m	0 h 35 m
Decl.	...	0° 22′ N	1° 46′ N	2° 9′ N

Saturn. Saturn is in Aries throughout and its motion direct till it reaches the stationary point on 14th August and then its course is changed to retrograde. During the beginning of June it is visible as a morning object and by the end of August it rises by 8 P.M.

The planet is in conjunction with the moon on 13th June, 11th July and 7th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	1 h 44 m	1 h 54 m	2 h 0 m
Decl.	...	8° 15′N	9° 6′N	9° 29′ N

Uranus. Uranus appears as a morning object on the Eastern sky. It is stationary on the 28th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	3 h 7 m	3 h 13 m	3 h 17 m
Decl.	...	17° 12′ N	17° 36′ N	17° 52′ N

Neptune. This planet rises just an hour after midnight during June and is visible throughout the night by the end of August.

It is stationary on the 2nd June.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	11 h 27 m	11 h 28 m	11 h 30 m
Decl.	...	4° 50′ N	4° 44′ N	4° 27′ N

H. SUBRAMANI IYER.

GLEANINGS

At least one of the great mechanical engineers of the cosmos seems to have convinced himself that the models were real things. It was that persistent confusion of the map with the thing mapped, and it was a perfect, hideous picture of the age which brought it forth.

Optimism hurdled all previous records; the riddle of the universe was about to be solved — by machinery. One more gadget here, a nut or two there, and the gigantic machine would continue to function till God or the second law of thermodynamics stopped it.

Then some careless mechanic dropped a monkey wrench. There was a stripping of gears, a last terrific threshing of broken piston rods, a mad scurrying of teetering wheels into the infinite, and the whole machine went to complete and final smash hortly safter the year 1900.

From E. T. BELL, *Numerology*.

REVIEWS

RAYMOND CLARE ARCHIBALD: *Outline of the History of Mathematics.* Fourth Edition, revised and enlarged. Price 50 cents. 1939. (The Mathematical Association of America, Inc., Oberlin, Ohio).

The need for propagating a knowledge of the History of Mathematics among students is now fairly well recognised. For teachers and research-workers, the knowledge is almost indispensable. The history, however, is not merely a collection of theorems chronologically arranged. It should give a satisfactory account of the men (or women), behind the theorems and also how they fit in with the period and the country in which they lived. For a real understanding of the spirit and scope of mathematics, its continuity, its inevitability and grandeur, the history of the subject is essential. Since the days of Ball's History of Mathematics, several historico-mathematical researches in Greek, Babylonian and Indian mathematics, as well as the immense growth of the subject itself in recent decades have added to the wealth of available material, from which only an imperfect selection can be made by any one trying to give an outline such as the one under review. In its original form the outline contained the substance of two lectures delivered by Prof. R. C. Archibald before Engineering teachers about eight years ago. It is evident that it has been a fairly popular book all these years and we congratulate the author on the present edition, which is an appreciable extension of the earlier editions with a number of changes and corrections due notably to the influence of Otto Neugebauer, the famous interpreter of Babylonian Mathematics.

The sages of ancient India condensed all important lore in the form of *Sutras* to facilitate their learning and retention. The same object is now achieved through Primers and Outlines. The present Outline is an effective summary of the History of Mathematics in two sections of about 25 pages each. The first section adopts the national basis and deals with Babylonian and Egyptian mathematics, Greek mathematics, Asiatic mathematics (the mathematics of the Hindu, Arabic, and Persian but not Chinese and Japanese who are now more interested in mutual destruction than in mathematics), and European mathematics of the period 1200–1600. By way of detail, I may mention that Hindu mathematics is finished off in 36 lines

with the mention of two names Brahmagupta and Bhaskara. Following probably T. L. Heath, the author attributes the general solution of $x^2 - Ay^2 = 1$ in integers, wrongly to Brahmagupta also. (For a proper idea of the Indian method, and its implications vide *Current Science*, Vol. VI, No. 12, June 1938, p. 602).

The second section deals with the history chronologically by centuries, stopping short however of the twentieth. Here the author has imposed upon himself certain limitations in regard to the subject-matter in order to sustain the interest of the general reader.

The book concludes with a valuable literature list and notes. This list is not, however, exhaustive, as there are many important omissions. We commend the book heartily to all students of mathematics. May I end this review with the enthusiastic utterance of the Vedic mathematician, more than 3000 years ago, in the following terms:

'Yathā sikhā mayurāṇām nāgānām maṇayo yathā
Tadvad vedānga Sastraṇām gaṇitam mūrdhani sthitam.'

'Even as the crests are to peacocks, and the hood-gems to serpents, so the Science of computation stands at the head of scientific accessories to the Vedas.'

The history of a science which has claimed such enthusiastic devotees from the very beginning of its existence ought to have a great cultural value.

A. A. K.

GEORGES HOSTELET. Le Progres de L'esprit II. *Les fondements expérimentaux de l'analyse mathématique des faits statistiques.* Pp. 70. 15fr. (1937). III. *Le concours de l'analyse mathématique a l'analyse expérimentale des faits statistiques.* Pp. 69. 15fr. (1938) Actualités scientifiques et industrielles, 552, 585. (Hermann, Paris).

The first of these pamphlets is a development from a communication on the experimental foundations of the mathematical analysis of statistical facts presented to the International Institute of Statistics at the Athens session. It also gives the author's replies to the comments raised during the session. The main point emphasized is that it is indispensable for the algebraists to elucidate clearly the experimental postulates underlying the formulæ offered by them for the benefit of the statistician, such as the arithmetic

mean, the standard deviation and the coefficient of binomial correlation. The statistician should be informed of the limiting conditions for the experimental validity of the mathematical formulæ and therefore of the conditions to be realised by the statistical facts in order that the formulæ may be legitimately applied to them. The author examines the experimental foundations of probability and chance, as well as Pearson's theory of contingency, which is deeply criticised as lacking in experimental validity.

The second pamphlet attempts to complete the train of thought pursued in the first, by elaborating the implications of the three modes of scientific investigation, viz., experimental empiricism, abstract deduction, and experimental abstraction. In the second part the author embarks on the discussion of the positions of M. Reichenbach, M. de Broglie, and M. Barzin towards the notions of causality and determinism, the comparison of philosophic and scientific attitudes and the role of scientific methodology.

In the view of the author's methodology, the experimental method may be said to be centripetal and the mathematical method centrifugal. Scientific exploration is a collaboration of the activities of intuition and reasoning controlled by experimental analysis.

Intuition by itself is fallible and therefore needs control, it is confused and therefore should be made more precise. The scientific method consists essentially in a certain mode of elaboration and verification of intuitions which spring in gifted minds in the course of their explorations of physical, biological, psychological and social facts. The interpretation of facts progresses, beginning from intuition, by experimental analysis controlled by legitimate abstractions and successive re-integrations of real conditions.

The papers are well worth the attention of all those who are interested in the scientific methodology of the modern continental school, of which one of the illustrious exponents is M. L. Brunschvicg, the editor of the series 'Le Progres de l'Esprit'.

Mysore. A. A. K.

BRIJ MOHAN: *The Intermediate Solid Geometry*, Mohan & Co., Moradabad. Pp. 120, Price Re. 1-4-0.

In this book which is designed to meet the requirements of the Intermediate in Arts students of Indian Universities, the author has presented in a concise form the contents of the first 21 propositions

of Euclid's Book XI. Among solids the Prism, Pyramid and general Polyhedra are treated and also the mensuration of the Cylinder, Cone and Sphere. Examples from various University papers are interspersed throughout the book, but one feels that the total number of exercises for the learner might have been increased with advantage. The matter is carefully arranged and presented and the diagrams and printing are fairly good.

K. RANGASWAMI.

JAHNKE-EMDE: *Functionentafeln mit Formeln und Kurven: Tables of Functions with Formulae and Curves*, Third revised edition, B. G. Teubner, 1938, pp. 305.

The second edition of this book (in two languages German and English) which is both a compendium of important formulae and a table of functions, appeared in 1933 and was reviewed in detail in *The Mathematics Student* Vol. III, pages 76-77. The most valuable feature of that edition was the graphical representation of functions by means of "relief surfaces"—a method which gives a wonderful insight into the characteristic features of the function as a whole. This is naturally retained in this edition and extended to the study of other functions such as the Mathieu functions for which the x, q surfaces are given for $ce_0(x, q)$, $ce_m(x, q)$, $se_m(x, q)$ $m=1, 2$.

The new functions included in this edition are the confluent hypergeometric function, the Mathieu functions associated with the elliptic cylinder, and the Lommel-Weber and Struve Functions of orders zero and one. The elaborate treatment of Bessel functions is retained and improved by bringing the Debye series into a more convenient form. In the complete elliptic integrals of the first and second kind, formulae and tables have been added for other than Legendre's standard form. On the other hand, the material of the first 75 pages of the old edition dealing with tables of powers, complex numbers, cubic equations and the elementary transcendental functions have been omitted. It is proposed to amplify this material and issue it in separate book form as there must be many who will use this part without being interested in the higher functions.

The opportunity afforded by the new edition has been utilised to being the references to the literature on the several functions up to date.

A. N. RAO.

BOOKS RECEIVED FOR REVIEW

Actualites Scientifiques et Industrielles: Hermann & Co., Paris.

No. 649. SCHWERDTFEGER: *Les fonctions de matrices.*

No. 650. CHAIT: *Essai d' explication des crises econometriques par l' econometrie.*

No. 657. BIERNACKI: *Les fonctions multivalentes.*

No. 660. VASILESCO: *La notion de point irrégulier dans le probleme de Dirichlet.*

No. 733. MANDELBROJT: *La rêgularisation des fonctions.*

No. 734-740. Colloque consacre a la Théorie des probabilités et presidé par M. Maurice Frechet.

No. 734. Conférences d' introduction.

No. 735. Les fondements du calcul des probabilités.

No. 736. Les sommes et les fonctions de variables aléatoires.

No. 737. Le principle ergodique et les probabilités en chaîne.

No. 738. Les fonctions aléatoires.

No. 739. Conceptions diverses.

No. 740. La statistique mathématique.

S. R. Gupta: Elementary Analytical Dynamics of a Particle, Atma Ram and Sons, Lahore, 1939.

ANNOUNCEMENTS AND NEWS

Mr. S. Subramaniam, M.A., Statistician to the Economic Adviser to the Government of India, Delhi, has been elected a member of the Society.

Arrangements for the Hyderabad Conference of the Indian Mathematical Society are in full progress. Dr. N. R. Sen, Ghosh Professor of Applied Mathematics, Calcutta University, will deliver the Inaugural Address.

Prof. D. D. Kosambi of Fergusson College will open the Symposium on Generalised Geometry and Dr. S. S. Pillai the symposium on Waring's Problem.

Popular lectures will be delivered by (1) Prof. M. Abdur Rahman Khan of Hyderabad (Deccan) on an Astronomical subject, (2) Prof. A. Narasinga Rao of Annamalainagar on Mathematical Puzzles and Amusements, and (3) Dr. T. Vijayaraghavan of Dacca.

The Osmania University hopes to be able to provide boarding and lodging for all delegates free of charge.

Papers intended for the conference should be sent (with 2 brief abstracts) to Prof. A. Narasinga Rao, Annamalainagar, before the end of October.

Dr. B. Ramamurti, M.A., D. Sc. (Mad.), lecturer in Mathematics, Annamalai University, has been appointed Professor and Head of the Mathematics Department, Government College, Ajmere.

Dr. V. Ganapati Iyer, M.A., D.Sc. (Mad.), and Mr. V. Seetharaman, M.A., M.Sc. (Annamalai), have been appointed Lecturers in Mathematics, Annamalai University, Annamalainagar.

The Semicentennial Celebration of the American Mathematical Society

We offer our hearty felicitations and good wishes to the American Mathematical Society which celebrated its Golden Jubilee at its birthplace, Columbia University, New York City, during September 1938. The main feature of the meeting was a series of ten invited addresses of which three by Profs. R. C. Archibald, G. D. Birkhoff and E. T. Bell dealt with the history of the Society and the development of mathematics in America, and the other seven by Profs. G. C. Evans, E. J. McShane, J. F. Ritt, J. L. Synge, T. Y. Thomas, Norbert Weiner and R. L. Wilder with specific mathematical topics. These, after amplification, have been published in two volumes under the general heading of Semicentennial Addresses. There were also valuable collections of manuscripts, rare editions, instruments, models, charts etc. on exhibit. The Founder of the Society Prof. Thomas Scott Fiske was presented, after the Gala dinner held in the Hotel Astor, with a beautifully illuminated testimonial containing a greeting of appreciation and affection prepared by order of the council and signed by the President and the Secretary.

The Nobel Prize in Physics has been awarded to Prof. Enrico Fermi of the University of Rome.

Prof. James Pierpont, since 1933 Professor Emeritus of Mathematics at Yale University and well known author of books on the Theory of Functions, died in December 1938.

SOLUTIONS TO QUESTIONS

Question 1768

(R. RAMANUJAM):—Construct geometrically a triangle given the lengths of the internal bisectors of the triangle.

Remarks by Johannes Mahrenholz

The problem depends on an equation of degree 10 and is not soluble by means of ruler and compasses. See the following references :—

(1) Terquem, Considérations sur le triangle rectilgne, *Nouvelles Annales de Mathematiques*, 1 (1842) p. 87.

(2) V. Reuthe Fink, *Crelle's Journal*, 26, p. 273.

(3) Bützberger, Ein mit der Theorie algebraischer Flächen zusammenhàngendes planimetrisches Problem. *Dissertation*, Bern, 1889.

(4) F. T. Van der Berg, *Nieuv Archief voor Wiskunde*, 16 (1889), p. 179.

(5) P. Barbarin, *Mathesis*, 16 (1896), p. 143.

(6) Korselt, Über die unmöglichkeit der Konstruktion eines Dreiecks aus der drei Winkelhalbierenden, *Zeitschrift für Mathematischen und naturwissenschaftlischen Unterricht*, 28 (1897), p. 81.

(7) Heymann, Zum problem der Winkelhalbierenden, *Ibid* 28 (1897), p. 165.

(8) Korselt, Über das Problem der Winkelhalbierenden, *Zeitschrift für Mathematik und Physik*, 42 (1897).

(9) Schmidt, Das problem der Winkelhalbierenden, *Jahresberricht Kotummal* Oberrealschule, Eger, 1902, 1903, 1904.

(10) Schmidt J. Über eine für die Dreieiksgeometrie wichtige Gleichung sechsten Grades, *Archiv für Mathematik und Physik*, 3 Recte, 13 (1908), p. 108.

The Question is also found as problems in :—

(11) *Rivista Matemetica* of Peano, 2, p. 34 (Rosotti).

(12) *Nouvelle correspondence de Mathematique*, 1 (1875), p. 208 and 3 (1877), p. 32 (Brocard).

(13) *Zeitschrift für mathematischen und naturwissenschaftlischen Unterricht*, 41 (1910), p. 218 (Degel).

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the official Journal.

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All communications intended for publication should be sent to the Editor, A. NARASINGA RAO, Annamalai University, Annamalainagar, South India. Communications relating to the receipt of the Journal, changes of address, and the purchase of current or back numbers of this journal should be sent to the Assistant Secretary, S. MAHADEVA IYER, Assistant Professor of Mathematics, Presidency College, Madras.

All communications regarding advertisements must be addressed to the Editor.

Remittances of membership fees and of subscriptions should be sent to the Treasurer, Prof. L. N. SUBRAMANIAM, Department of Mathematics, Christian College, Tambaram, Madras Presidency.